

# Looijenga's weighted projective space, Tate's algorithm and Mordell-Weil Lattice in F-theory and heterotic string theory

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## Abstract

It is now well known that the moduli space of a vector bundle for heterotic string compactifications to four dimensions is parameterized by a set of sections of a weighted projective space bundle of a particular kind, known as Looijenga's weighted projective space bundle. We show that the requisite weighted projective spaces and the Weierstrass equations describing the spectral covers for gauge groups  $E_N$  ( $N = 4, \dots, 8$ ) and  $SU(n+1)$  ( $n = 1, 2, 3$ ) can be obtained systematically by a series of blowing-up procedures according to Tate's algorithm, thereby the sections of correct line bundles claimed to arise by Looijenga's theorem can be automatically obtained. They are nothing but the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory parameterizing the complex structure, which is further confirmed in the constructions of  $D_4$ ,  $A_5$ ,  $D_6$ ,  $E_3$  and  $SU(2) \times SU(2)$  bundles. We also explain why we can obtain them in this way by using the structure theorem of the Mordell-Weil lattice, which is also useful for understanding the relation between the singularity and the occurrence of chiral matter in F-theory.

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## I. INTRODUCTION

It is now a well-known fact that holomorphic vector bundles on an elliptically fibered Calabi-Yau, needed for heterotic string compactifications to four dimensions, are constructed by considering spectral covers [1]. A spectral cover is basically a ramified  $n$ -fold cover (for an  $SU(n)$  bundle) of the base of the elliptic Calabi-Yau, representing the Wilson lines in each elliptic fiber as points on the elliptic fiber identified with its dual. One then introduces a twisting line bundle over the base whose first Chern class (the  $\eta$  class) is related to the number of instantons of the bundle. Once one has a spectral surface and a line bundle over it, one can construct a vector bundle via the Fourier-Mukai transform using the Poincare bundle, up to a so-called  $\gamma$  class corresponding to the  $G$ -flux in the F-theory dual. For more detail of the spectral cover construction, see *e.g.* [1–4].

It is also now well known that the moduli space of the vector bundle is parameterized by a set of sections of a weighted projective space bundle of a particular kind, known as Looijenga’s weighted projective space bundle. Some time ago, for  $E_8$ ,  $E_7$  and  $E_6$  bundles and other lower-rank ones Looijenga’s theorem was confirmed [1] (except for some subtlety in  $E_8$ ) by explicitly constructing spectral covers by using del Pezzo surfaces. Although this approach was enough to reveal the validity of the miraculous nature of Looijenga’s theorem, the constructions of the bundles were done case by case and appear to be independent and unrelated with each other. In this paper we will show that the requisite weighted projective spaces and the Weierstrass equations describing the spectral covers for  $E_8$  through  $A_1$  bundles can be obtained systematically by a series of blowing up procedures according to the well-known Tate’s algorithm, thereby the sections of correct line bundles claimed to arise by the theorem can be automatically obtained. We will also explain why we can obtain them in this way by using the structure theorem of the Mordell-Weil lattice [5].

We will also show that the structure theorem of the Mordell-Weil lattice is useful for understanding the relation between the singularity and the occurrence of chiral matter in F-theory. (This new role of the Mordell-Weil lattice in F-theory was already briefly discussed in [6].) In the literature the relation between sections and the appearance of chiral matter is somewhat indirect. That is, on the heterotic side one considers a vector bundle on a spectral cover and computes the cohomology by means of the Leray spectral sequence to find that the chiral matter is localized where one or some of the “matter curves” representing the Wilson

lines go(es) to infinity (zero in the addition rule of the  $\wp$  function). On the F-theory side, the del Pezzo surface (or rational elliptic surface) itself in which the spectral cover is defined becomes the fiber with an appropriate twist corresponding to the weighted projective space bundle of Looijenga, and matter arise where the singularity is enhanced [7–9] along the 7-brane. We will show that the structure of the Mordell-Weil lattice ensures the compatibility of these two pictures of chiral matter generations.

The plan of this paper is as follows. In section 2, we start from the same degree six equation in the weighted projective space  $\mathbb{WP}_{(1,1,2,3)}^3$  given in [1] for  $E_8$  bundles and review the construction of the spectral cover. Then we tune some of the sections to be in a special form so that the  $dP_9$  develops a singularity. It turns out that, by blowing up the singularity, we are automatically led to the equation for  $E_7$  bundles discussed in [1], where the relevant sections are precisely the ones constituting the correct weighted projective space of Looijenga. Repeating a similar procedure, we will find a series of spectral covers for the vector bundles from  $E_7$  through  $A_1$ . In section 3 we will see that the sections parameterizing a Looijenga’s weighted projective space are nothing but the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory parameterizing the complex structure of the elliptic manifold with a singularity orthogonal to the gauge group of the vector bundle in the whole  $E_8$ . This fact is further confirmed in the constructions of  $D_4$ ,  $A_5$ ,  $D_6$ ,  $E_3$  and  $SU(2) \times SU(2)$  bundles. In section 4 we discuss why this is possible by introducing the structure theorem of the Mordell-Weil lattice. The final section is devoted to conclusions.

## II. TATE’S ALGORITHM AND LOOIJENGA’S WEIGHTED PROJECTIVE SPACE

### A. $E_8$ bundles: the generic case

We start with the construction of  $E_8$  bundles, following [1]. As pointed out there, it is well known that  $E_8$  bundles have exceptional features but the construction is important because it is the starting point of all the constructions of the vector bundles for other gauge groups of lower ranks.

Let us first consider a generic degree six equation in  $\mathbb{WP}_{(1,1,2,3)}^3$  [1] with homogeneous

coordinates  $(u, v, X, Y) \sim (\lambda u, \lambda v, \lambda^2 X, \lambda^3 Y)$  ( $\lambda \in \mathbf{C}$ ):

$$\begin{aligned} 0 = & Y^2 + X^3 - \frac{g_2}{4}Xv^4 - \frac{g_3}{4}v^6 \\ & +(\beta_4u^4 + \beta_3u^3v + \beta_2u^2v^2 + \beta_1uv^3)X \\ & +(\alpha_6u^6 + \alpha_5u^5v + \alpha_4u^4v^2 + \alpha_3u^3v^3 + \alpha_1uv^5). \end{aligned} \quad (1)$$

In a patch  $u \neq 0$ , we define affine coordinates  $(z, x, y)$  by  $(u, v, X, Y) \sim (1, \frac{v}{u}, \frac{X}{u^2}, \frac{Y}{u^3}) \equiv (1, z, x, y)$ . Then we have

$$\begin{aligned} 0 = & y^2 + x^3 - \frac{g_2}{4}xz^4 - \frac{g_3}{4}z^6 \\ & +(\beta_4 + \beta_3z + \beta_2z^2 + \beta_1z^3)x \\ & +(\alpha_6 + \alpha_5z + \alpha_4z^2 + \alpha_3z^3 + \alpha_1z^5). \end{aligned} \quad (2)$$

(1) is a  $dP_8$ , and by blowing up at  $u = v = 0$  it becomes a rational elliptic surface  $dP_9$  with section [1]. Then it can be viewed as an elliptic fibration over  $\mathbb{P}^1$  whose coordinates are  $(u : v)$ , the affine coordinate being  $z$  in the affine patch  $u \neq 0$ .

To serve as a part of compactifications of F-theory [10] to lower dimensions, this  $\mathbb{P}^1$  must be further fibered over some base space  $\mathcal{B}$ , where the coefficients  $\alpha_i$  ( $1, 3, \dots, 6$ ) and  $\beta_j$  ( $j = 1, \dots, 4$ ) as well as the coordinates are promoted to sections of some appropriate line bundles over the  $\mathbb{P}^1$  fibration. More precisely, we regard this  $dP_9$  as a part of an elliptic  $K3$  in the stable degeneration limit, which itself is fibered over  $\mathcal{B}$  in such a way that the total space is an elliptic Calabi-Yau  $\mathcal{Y}$ , whose base  $\mathcal{W}$  itself is a  $\mathbb{P}^1$  fibration over  $\mathcal{B}$ . We take

$$\begin{aligned} X &\in \Gamma((\mathcal{L} \otimes \mathcal{N})^2), \\ Y &\in \Gamma((\mathcal{L} \otimes \mathcal{N})^3), \\ v &\in \Gamma(\mathcal{N}), \\ u &\in \Gamma(\mathcal{L}^6), \end{aligned} \quad (3)$$

where  $\Gamma$  denotes the space of the sections,  $\mathcal{L}$  is the anti-canonical line bundle of the base  $\mathcal{B}$ , and  $\mathcal{N}$  is the “twisting” line bundle over  $\mathcal{B}^1$  characterizing the vector bundle of the dual heterotic string theory compactified on an elliptic Calabi-Yau  $\mathcal{Z}$  of complex dimension one less, whose complex structure is identical to that of the elliptic fibration at  $z = \infty$ . At

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<sup>1</sup> The first Chern class of  $\mathcal{N}$  is customarily referred to as “the  $\eta$  class.”

the same time, the fiber of this elliptic fibration at infinity also plays the role of the “dual” torus, at which the values of rational sections of (1) describe the spectral surface [1], *i.e.* the holonomies of the flat connections, and hence the moduli space of the heterotic vector bundle  $V^2$ . Then the affine coordinates  $(z, x, y)$  transform as

$$\begin{aligned} z &\in \Gamma(\mathcal{M}), \\ x &\in \Gamma((\mathcal{L} \otimes \mathcal{M})^2), \\ y &\in \Gamma((\mathcal{L} \otimes \mathcal{M})^3) \end{aligned} \tag{4}$$

as sections of line bundles over  $\mathcal{B}$ , where  $\mathcal{M} \equiv \mathcal{L}^{-6} \otimes \mathcal{N}$ . They are also sections of some line bundles over the  $\mathbb{P}^1$  with  $(u : v)$  being its coordinates. Due to the Calabi-Yau condition for  $\mathcal{Y}$ ,  $\mathcal{W}$  is required to be such that the base  $\mathcal{B}$  part of the anti-canonical class of  $\mathcal{W}$  coincides with  $\mathcal{L} \otimes \mathcal{M}$  (4).

For example, if we take  $\mathcal{B}$  to be  $\mathbb{P}^1$  and  $\mathcal{Z}$  to be an elliptic  $K3$ , then  $\mathcal{L}$  is an  $\mathcal{O}(2)$  bundle whose sections are described by quadratic polynomials of the affine coordinate  $z'$  of the base  $\mathbb{P}^1$ . Also  $\mathcal{N}$  is chosen to be an  $\mathcal{O}(12 + n) = \mathcal{L}^6 \otimes \mathcal{L}^{\frac{n}{2}}$  bundle, corresponding to  $12 + n$  instantons for one of the two  $E_8$  gauge groups of the six-dimensional heterotic string theory.  $\mathcal{M}$  is  $\mathcal{L}^{\frac{n}{2}}$ . The corresponding dual F-theory description chooses [7]  $\mathcal{W}$  to be a Hirzebruch surface  $\mathbb{F}_n$  so that the base  $\mathbb{P}^1$  part of the anti-canonical class is  $\mathcal{L} \otimes \mathcal{M} = \mathcal{O}(2 + n)$  (or  $\mathcal{O}(2 - n)$  depending on the choice of the divisor “at infinity”), geometrically realizing the twisting of the spectral cover of the heterotic dual. Then we are led to the well-known Weierstrass equation on a Hirzebruch surface [7]

$$0 = y^2 + x^3 + f(z, z')x + g(z, z'), \tag{5}$$

$$\begin{aligned} f(z, z') &= f_{8+4n} + zf_{8+3n} + z^2f_{8+2n} + z^3f_{8+n} + \cdots, \\ g(z, z') &= g_{12+6n} + zg_{12+5n} + z^2g_{12+4n} + z^3g_{12+3n} + \cdots, \end{aligned} \tag{6}$$

where  $f_{8+(4-i)n}$  and  $g_{12+(6-j)n}$  ( $i, j = 0, 1, 2, \dots$ ) are polynomials of  $z'$  with subscripts being their degrees in  $z'$ .<sup>3</sup>

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<sup>2</sup> We assume  $c_1(V) = 0$  throughout this paper.

<sup>3</sup> In the *four*-dimensional compactifications of F-theory, one also needs to specify the so-called  $\gamma$  class ( $G$ -flux), but it is irrelevant for the discussion here.

In the general case, we write (1) in the Neron-Tate's form:

$$\begin{aligned}
0 = & y^2 + x^3 + (a_{1,0} + a_{1,1}z + a_{1,2}z^2 + \cdots)xy \\
& + (a_{2,0} + a_{2,1}z + a_{2,2}z^2 + \cdots)x^2 \\
& + (a_{3,0} + a_{3,1}z + a_{3,2}z^2 + \cdots)y \\
& + (a_{4,0} + a_{4,1}z + a_{4,2}z^2 + \cdots)x \\
& + a_{6,0} + a_{6,1}z + a_{6,2}z^2 + \cdots.
\end{aligned} \tag{7}$$

The coefficients must be

$$\begin{aligned}
f_{8+4n} = \beta_4 = a_{4,0} & \in \Gamma(\mathcal{L}^{-20} \otimes \mathcal{N}^4) = \Gamma(\mathcal{L}^4 \otimes \mathcal{M}^4) \\
f_{8+3n} = \beta_3 = a_{4,1} & \in \Gamma(\mathcal{L}^{-14} \otimes \mathcal{N}^3) = \Gamma(\mathcal{L}^4 \otimes \mathcal{M}^3) \\
f_{8+2n} = \beta_2 = a_{4,2} & \in \Gamma(\mathcal{L}^{-8} \otimes \mathcal{N}^2) = \Gamma(\mathcal{L}^4 \otimes \mathcal{M}^2) \\
f_{8+n} = \beta_1 = a_{4,3} & \in \Gamma(\mathcal{L}^{-2} \otimes \mathcal{N}^1) = \Gamma(\mathcal{L}^4 \otimes \mathcal{M}^1) \\
\\
g_{12+6n} = \alpha_6 = a_{6,0} & \in \Gamma(\mathcal{L}^{-30} \otimes \mathcal{N}^6) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^6) \\
g_{12+5n} = \alpha_5 = a_{6,1} & \in \Gamma(\mathcal{L}^{-24} \otimes \mathcal{N}^5) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^5) \\
g_{12+4n} = \alpha_4 = a_{6,2} & \in \Gamma(\mathcal{L}^{-18} \otimes \mathcal{N}^4) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^4) \\
g_{12+3n} = \alpha_3 = a_{6,3} & \in \Gamma(\mathcal{L}^{-12} \otimes \mathcal{N}^3) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^3) \\
(g_{12+2n} = \alpha_2 = a_{6,4} & \in \Gamma(\mathcal{L}^{-6} \otimes \mathcal{N}^2) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^2)) \\
g_{12+n} = \alpha_1 = a_{6,5} & \in \Gamma(\mathcal{L}^{-0} \otimes \mathcal{N}^1) = \Gamma(\mathcal{L}^6 \otimes \mathcal{M}^1).
\end{aligned} \tag{8}$$

In the above equations we have also displayed on the leftmost side the corresponding coefficient polynomials of the Weierstrass form for the well-studied six-dimensional compactification.

Now Looijenga's theorem states that the moduli space of the vector bundle is parameterized by the sections

$$a_k = \Gamma(\mathcal{L}^{-d_k} \otimes \mathcal{N}^{s_k}) \quad (k = 0, \dots, \text{rank} G), \tag{9}$$

where  $d_k$  is 0 ( $k = 0$ ) or the degree of the independent Casimir of  $G$ , and  $s_k$  is 1 ( $k = 0$ ) or the coefficient of the  $k$ -th coroot when the lowest root  $-\theta$  is expanded.

In the present case, the minus of the powers of  $\mathcal{L}$  in the middle column of (8) read

$$0, 2, 8, 12, 14, 18, 20, 24, 30, \tag{10}$$

which precisely coincide with (“0” and) the set of degrees of independent Casimirs of  $E_8$ , while the powers of  $\mathcal{N}$  are *close to* identical to the coefficients of the (co)root ( $E_8$  is simply laced) expansion:

$$-\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8, \quad (11)$$

except that the power for  $g_{12+n} = \alpha_1 = a_{6,5}$ , which should be 2, is 1. If  $g_{12+2n} = \alpha_2 = a_{6,4}$  were taken instead, then the power would become 2 which is correct, but then the power of  $\mathcal{L}^{-1}$  would be 6 which does not agree with Looijenga’s statement. Thus in this  $E_8$  case, we have obtained the weighted projective space  $\mathbb{WP}_{(1,2,2,3,3,4,4,5,6)}^8$  but (9) is not completely true [1].

### B. $E_7$ bundles by blowing up ( $A_1$ singularity)

In [1] it was shown that  $E_7$  bundles can be constructed in terms of a degree-4 equation in  $\mathbb{WP}_{(1,1,1,2)}^3$ . Sections of Looijenga’s weighted projective bundle are similarly parameterized by the coefficients of the Weierstrass equation, which themselves are sections of a particular set of line bundles specified by Looijenga’s theorem (9). In this section we will show that these setups naturally arise by blowing up the singularity on the degree-6 equation in  $\mathbb{WP}_{(1,1,2,3)}^3$  for  $E_8$  discussed in the previous section, according to a well-known procedure known as Tate’s algorithm.

Physically, an  $E_7$  bundle implies an  $SU(2)$  unbroken gauge symmetry of one of the two  $E_8$  of heterotic string theory. Mathematically, this is a reflection of the structure of the Mordell-Weil lattice [5] stating the complementarity in  $E_8$  of the sections and the singularities of a rational elliptic surface  $dP_9$ .

Below we use in the process of blowing up, *even in the general case not restricted to the six-dimensional case*, the notation:

$$f_{8+(4-i)n} := a_{4,i}, \quad (12)$$

$$g_{12+(6-j)n} := a_{6,j}, \quad (13)$$

by using the dictionary (8).<sup>4</sup> In that case,  $n$  no longer has the meaning of the number of

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<sup>4</sup> More generally, a degree  $(an + b)$  polynomial in  $z'$  in the 6D F-theory compactification corresponds to a section of  $\mathcal{L}^{-d} \otimes \mathcal{N}^s$  with  $d = 6a - \frac{b}{2}$  and  $s = a$ .



instantons, but is rather just a dummy variable with its coefficient specifying the powers of the twisting line bundle to which the section belong. This notation is intended for the convenience of, and will be particularly useful to, the readers who are familiar with the well-known six-dimensional F-theory compactification [7, 8]. This enables us to easily recognize that the sections parameterizing a Looijenga's weighted projective space are nothing but the set of independent polynomials parameterizing the complex structure of the elliptic manifold in F-theory with a singularity, which is the orthogonal complement in  $E_8$  of the gauge group of the vector bundle. Why they are the orthogonal complement of each other will be explained in section 4.

In order to have an  $SU(2) = A_1 = I_2$  singularity, we assume that the coefficients  $f_{8+4n}$ ,  $g_{12+6n}$  and  $g_{12+5n}$  in (6) can be written in term of some  $h_{2n+4} \in \Gamma(\mathcal{L}^2 \otimes \mathcal{M}^2)$  as

$$\begin{aligned} f_{8+4n} &= -3h_{2n+4}^2, \\ g_{12+6n} &= 2h_{2n+4}^3, \\ g_{12+5n} &= -h_{2n+4}f_{8+3n}. \end{aligned} \tag{14}$$

Then the discriminant

$$\Delta = 4f(z, z')^3 + 27g(z, z')^2 \tag{15}$$

becomes  $O(z^2)$  or higher at  $z = 0$ , implying an  $A_1$  singularity. The location of the singularity is  $y = z = 0$  but  $x = h_{2n+4}$ , so it is not at the origin in general. So we define

$$x_{new} \equiv x - h_{2n+4}, \tag{16}$$

then (5) becomes

$$\begin{aligned} 0 &= y^2 + x_{new}^3 + 3h_{2n+4}x_{new}^2 + (f_{8+3n}z + f_{8+2n}z^2 + f_{8+n}z^3 + f_8z^4)x_{new} \\ &\quad + (h_{2n+4}f_{8+2n} + g_{12+4n})z^2 + (h_{2n+4}f_{8+n} + g_{12+3n})z^3 \\ &\quad + (h_{2n+4}f_8 + g_{12+2n})z^4 + g_{12+n}z^5 + g_{12}z^6. \end{aligned} \tag{17}$$

By construction it has a singularity at  $x_{new} = y = z = 0$  so we blow it up at  $(x_{new}, y, z) = (0, 0, 0) \in \mathbf{C}^3$  by defining

$$\tilde{\mathbf{C}}^3 = \{((x_{new}, y, z), (\xi : \eta : \zeta)) \in \mathbf{C}^3 \times \mathbb{P}^2 | (x_{new}, y, z) \in (\xi : \eta : \zeta)\}, \tag{18}$$

where  $(x_{new}, y, z) \in (\xi : \eta : \zeta)$  means that  $(x_{new}, y, z)$  and  $(\xi : \eta : \zeta)$  are parallel to each other.

Let  $x' \equiv \frac{\xi}{\zeta}$ ,  $y' \equiv \frac{\eta}{\zeta}$  in the affine patch with  $\zeta \neq 0$ , then

$$(x_{new}, y, z) = (x'z, y'z, z). \quad (19)$$

Plugging this into (17) and dividing it by  $z^2$ , we have

$$\begin{aligned} 0 = & y'^2 + x'^3 z + 3h_{2n+4}x'^2 + (f_{8+3n} + f_{8+2n}z + f_{8+n}z^2 + f_8z^3)x' \\ & + (h_{2n+4}f_{8+2n} + g_{12+4n}) + (h_{2n+4}f_{8+n} + g_{12+3n})z \\ & + (h_{2n+4}f_8 + g_{12+2n})z^2 + g_{12+n}z^3 + g_{12}z^4. \end{aligned} \quad (20)$$

One can show that this is a smooth curve *unless*

$$f_{8+3n}^2 - 12h_{2n+4}(h_{2n+4}f_{8+2n} + g_{12+4n}) = 0 \quad (21)$$

is satisfied. In fact, the left hand side of (21) is the coefficient of the  $O(z^2)$  term of the discriminant  $\Delta$ , so that (21) implies  $\text{ord}(\Delta) \geq 3$  at  $z = 0$ . Note that this is a necessary condition for the curve to be singular, and even if (21) holds, (20) still remains regular unless some additional conditions are satisfied, as we see in the next section.

Now we observe that (20) is nothing but a degree-4 equation in  $\mathbb{WP}_{(1,1,1,2)}^3((u, v, X', Y') \sim (\lambda u, \lambda v, \lambda X', \lambda^2 Y'))$ :

$$\begin{aligned} 0 = & Y'^2 + X'^3 v + a_{2,0}X'^2 u^2 \\ & + (a_{4,1}u^3 + a_{4,2}u^2 v + a_{4,3}uv^2 + a_{4,4}v^3)X' \\ & + (a_{6,2}u^4 + a_{6,3}u^3 v + a_{6,4}u^2 v^2 + a_{6,5}uv^3 + a_{6,6}v^4) \end{aligned} \quad (22)$$

expressed in the affine patch with  $u \neq 0$  in terms of the affine coordinates  $(u, v, X', Y') \sim (1, \frac{v}{u}, \frac{X'}{u}, \frac{Y'}{u^2}) \equiv (1, z, x', y') :$

$$\begin{aligned} 0 = & y'^2 + x'^3 z + a_{2,0}x'^2 \\ & + (a_{4,1} + a_{4,2}z + a_{4,3}z^2 + a_{4,4}z^3)x' \\ & + (a_{6,2} + a_{6,3}z + a_{6,4}z^2 + a_{6,5}z^3 + a_{6,6}z^4) \end{aligned} \quad (23)$$

with

$$\begin{aligned}
a_{2,0} &= 3h_{2n+4} \\
a_{4,1} &= f_{8+3n} \\
a_{4,2} &= f_{8+2n} \\
a_{4,3} &= f_{8+n} \\
a_{4,4} &= f_8 \\
a_{6,2} &= h_{2n+4}f_{8+2n} + g_{12+4n} \\
a_{6,3} &= h_{2n+4}f_{8+n} + g_{12+3n} \\
a_{6,4} &= h_{2n+4}f_8 + g_{12+2n} \\
a_{6,5} &= g_{12+n} \\
a_{6,6} &= g_{12}.
\end{aligned} \tag{24}$$

The degree-4 equation in  $\mathbb{WP}_{(1,1,1,2)}^3$  (22) is precisely the one found in [1] for  $E_7$  bundles. Thus we see that the set up for the construction of  $E_7$  bundles in [1] is naturally obtained by blowing up the singularity of the Weierstrass equation in  $\mathbb{WP}_{(1,1,2,3)}^3$  for  $E_8$  bundles.

Since

$$a_{i,j} \in \mathcal{L}^i \otimes \mathcal{M}^{i-j} = \mathcal{L}^{6j-5i} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^{i-j}, \tag{25}$$

we have

$$\begin{aligned}
a_{2,0} &\in \Gamma(\mathcal{L}^{-10} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{4,1} &\in \Gamma(\mathcal{L}^{-14} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^3) \\
a_{4,2} &\in \Gamma(\mathcal{L}^{-8} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{6,2} &\in \Gamma(\mathcal{L}^{-18} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^4) \\
a_{6,3} &\in \Gamma(\mathcal{L}^{-12} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^3) \\
a_{6,4} &\in \Gamma(\mathcal{L}^{-6} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1).
\end{aligned} \tag{26}$$

Note that  $a_{4,4} \in \Gamma(\mathcal{L}^4)$  or  $a_{6,6} \in \Gamma(\mathcal{L}^6)$  does not have information of the vector bundles but describes the complex structure of the Calabi-Yau on the heterotic side, and that of the elliptic fibration connecting the two  $dP_9$  fibrations on the F-theory side.

We see in (26) that the minus of the powers of  $\mathcal{L}$  read

$$0, 2, 6, 8, 10, 12, 14, 18, \quad (27)$$

which coincides with the set of degrees of independent Casimirs of  $E_7$  (including 0), and the powers of  $\mathcal{N}$  are the coefficients of the expansion of the highest root of  $E_7$ :

$$-\theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 1\alpha_7, \quad (28)$$

being (this time) in complete agreement with Looijenga.

### C. $E_6$ bundles ( $A_2$ singularity)

Next we suppose that (21) is satisfied. Since the first term is a square of a section, so must be the second term. This is achieved by requiring

$$\begin{aligned} h_{2n+4} &= h_{n+2}^2 \\ f_{8+3n} &= 12h_{n+2}H_{2n+6} \\ g_{12+4n} &= 12H_{2n+6}^2 - h_{n+2}^2 f_{8+2n} \end{aligned} \quad (29)$$

for some  $h_{n+2}$  and  $H_{2n+6}$ . These conditions are the ones for the exceptional curve to factorize into two lines, and the singularity becomes  $I_3$  of the Kodaira classification. This is smooth unless

$$-2H_{2n+6}(h_{n+2}^2 f_{8+2n} + 4H_{2n+6}^2) + h_{n+2}^3(h_{n+2}^2 f_{8+n} + g_{12+3n}) = 0 \quad (30)$$

is satisfied, in which the order of the discriminant would become higher than 3 and we would need a further blow-up. Plugging (29) into (17), we find

$$\begin{aligned} 0 &= y^2 + x_{new}^3 + 3h_{n+2}^2 x_{new}^2 + (12h_{n+2}H_{2n+6}z + f_{8+2n}z^2 + f_{8+n}z^3 + f_8z^4)x_{new} \\ &\quad + 12H_{2n+6}^2 z^2 + (h_{n+2}^2 f_{8+n} + g_{12+3n})z^3 + (h_{n+2}^2 f_8 + g_{12+2n})z^4 + g_{12+n}z^5 + g_{12}z^6. \end{aligned} \quad (31)$$

We can further rewrite it in terms of

$$y_{new} \equiv y - \sqrt{3}i(h_{n+2}x_{new} + 2H_{2n+6}z) \quad (32)$$

as

$$0 = y_{new}^2 + x_{new}^3 + 2\sqrt{3}ih_{n+2}x_{new}y_{new} + 4\sqrt{3}iH_{2n+6}zy_{new} + (f_{8+2n}z^2 + f_{8+n}z^3 + f_8z^4)x_{new} \\ + (h_{n+2}^2f_{8+n} + g_{12+3n})z^3 + (h_{n+2}^2f_8 + g_{12+2n})z^4 + g_{12+n}z^5 + g_{12}z^6. \quad (33)$$

Note that  $x_{new} = y = 0 \Leftrightarrow x_{new} = y_{new} = 0$  at  $z = 0$ . Similarly to (19) in (17), we set

$$(x_{new}, y_{new}, z) = (x'z, y'z, z) \quad (34)$$

to find

$$0 = y'^2 + x'^3z + 2\sqrt{3}ih_{n+2}x'y' + 4\sqrt{3}iH_{2n+6}y' + (f_{8+2n}z + f_{8+n}z^2 + f_8z^3)x' \\ + (h_{n+2}^2f_{8+n} + g_{12+3n})z + (h_{n+2}^2f_8 + g_{12+2n})z^2 + g_{12+n}z^3 + g_{12}z^4. \quad (35)$$

Again, this is a fourth-order equation in  $W\mathbb{P}_{(1,1,1,2)}^3((u, v, X', Y') \sim (\lambda u, \lambda v, \lambda X', \lambda^2 Y'))$ :

$$0 = Y'^2 + X'^3v + a_{1,0}X'Y'u + a_{3,1}Y'u^2 \\ + (a_{4,2}u^2v + a_{4,3}uv^2 + a_{4,4}v^3)X' \\ + a_{6,3}u^3v + a_{6,4}u^2v^2 + a_{6,5}uv^3 + a_{6,6}v^4 \quad (36)$$

expressed in terms of the affine coordinates  $(u, v, X', Y') \sim (1, \frac{v}{u}, \frac{X'}{u}, \frac{Y'}{u^2}) \equiv (1, z, x', y')$  in the patch  $u \neq 0$ :

$$0 = y'^2 + x'^3z + a_{1,0}x'y' + a_{3,1}y' \\ + (a_{4,2}z + a_{4,3}z^2 + a_{4,4}z^3)X' \\ + a_{6,3}z + a_{6,4}z^2 + a_{6,5}z^3 + a_{6,6}z^4 \quad (37)$$

with

$$a_{1,0} = 2\sqrt{3}ih_{n+2} \\ a_{3,1} = 4\sqrt{3}iH_{2n+6} \\ a_{4,2} = f_{8+2n} \\ a_{4,3} = f_{8+n} \\ a_{4,4} = f_8 \\ a_{6,3} = h_{2n+4}f_{8+n} + g_{12+3n} \\ a_{6,4} = h_{2n+4}f_8 + g_{12+2n} \\ a_{6,5} = g_{12+n} \\ a_{6,6} = g_{12}. \quad (38)$$

In this case we have

$$\begin{aligned}
a_{1,0} &\in \Gamma(\mathcal{L}^{-5} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{3,1} &\in \Gamma(\mathcal{L}^{-9} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{4,2} &\in \Gamma(\mathcal{L}^{-8} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{6,3} &\in \Gamma(\mathcal{L}^{-12} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^3) \\
a_{6,4} &\in \Gamma(\mathcal{L}^{-6} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1),
\end{aligned} \tag{39}$$

which is consistent with the facts that the degrees of the independent Casimirs of  $E_6$  (including 0) are

$$0, 2, 5, 6, 8, 9, 12 \tag{40}$$

and the (co)root expansion of the highest root is

$$-\theta = 1\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 1\alpha_6. \tag{41}$$

Thus we have seen that not only  $E_7$  bundles but  $E_6$  bundles can also be constructed from  $\mathbb{WP}_{(1,1,1,2)}^3$ . In contrast, instead of  $\mathbb{WP}_{(1,1,1,2)}^3$ ,  $\mathbb{WP}_{(1,1,1,1)}^3$  was used in FMW [1], which can be obtained by a further blow-up as we will see in the next section. Note, however, that in the case of the  $I_3 = A_2$  singularity the exceptional curve arising in the  $I_2 = A_1$  singularity simply splits into two lines, in which no additional blow-ups are needed, and therefore  $\mathbb{WP}_{(1,1,1,2)}^3$  suffice. Of course, one is free to blow it up so it is not a contradiction.

#### D. $D_5$ bundles ( $A_3$ singularity)

In this section we consider the case in which (30) is satisfied and the curve in the previous section becomes singular. Then the discriminant  $\Delta$  becomes  $\text{ord}(\Delta) \geq 4$ . In this case we require

$$H_{2n+6} = h_{n+2}H_{n+4} \tag{42}$$

for some  $H_{n+4}$ . Due to (30), we need to have

$$g_{12+3n} = 2H_{n+4}(f_{8+2n} + 4H_{n+4}^2) - h_{n+2}^2 f_{8+n}. \tag{43}$$

Then (35) becomes singular at  $x' = -2H_{n+4}$ ,  $y' = z = 0$ . To resolve this singularity we define

$$x'_{new} \equiv x' + 2H_{n+4}, \quad (44)$$

then

$$\begin{aligned} 0 = & y'^2 + x'_{new}{}^3 z + 2i\sqrt{3}h_{n+2}x'_{new}y' - 6zH_{n+4}x'_{new}{}^2 \\ & + x'_{new} (z(f_{8+2n} + 12H_{n+4}^2) + f_{8+n}z^2 + f_8z^3) \\ & + z^2 (h_{n+2}^2 f_8 - 2H_{n+4}f_{8+n} + g_{12+2n}) \\ & + z^3 (-2H_{n+4}f_8 + g_{12+n}) + g_{12}z^4. \end{aligned} \quad (45)$$

The singularity is located at  $x'_{new} = y' = z = 0$ , so defining

$$y' = \tilde{y}'z \quad (46)$$

and factoring  $z$  out, we derive

$$\begin{aligned} 0 = & \tilde{y}'^2 z + x'_{new}{}^3 + f_8 x'_{new} z^2 + g_{12} z^3 \\ & + 2i\sqrt{3}h_{n+2}x'_{new}\tilde{y}' - 6H_{n+4}x'_{new}{}^2 \\ & + x'_{new} ((f_{8+2n} + 12H_{n+4}^2) + f_{8+n}z) \\ & + z (h_{n+2}^2 f_8 - 2H_{n+4}f_{8+n} + g_{12+2n}) \\ & + z^2 (-2H_{n+4}f_8 + g_{12+n}). \end{aligned} \quad (47)$$

Rewriting this equation as

$$\begin{aligned} 0 = & \tilde{y}'^2 z + x'_{new}{}^3 + a_{4,4}x'_{new}z^2 + a_{6,6}z^3 \\ & + a_{1,0}x'_{new}\tilde{y}' + a_{2,1}x'_{new}{}^2 \\ & + x'_{new} (a_{4,2} + a_{4,3}z) \\ & + a_{6,4}z + a_{6,5}z^2, \end{aligned} \quad (48)$$

we see that this is a third-order equation in  $\mathbb{WP}_{(1,1,1,1)}^3 ((u, v, X', \tilde{Y}') \sim (\lambda u, \lambda v, \lambda X', \lambda \tilde{Y}'))$ :

$$\begin{aligned} 0 = & \tilde{Y}'^2 v + X'^3 + a_{4,4}X'v^2 + a_{6,6}v^3 \\ & + u^2 (a_{4,2}X' + a_{6,4}v) \\ & + u (a_{1,0}X'\tilde{Y}' + a_{2,1}X'^2 + a_{4,3}X'v + a_{6,5}v^2), \end{aligned} \quad (49)$$

expressed in terms of the affine coordinates  $(u, v, X', \tilde{Y}') \sim (1, \frac{v}{u}, \frac{X'}{u}, \frac{\tilde{Y}'}{u}) \equiv (1, z, x'_{new}, \tilde{y}')$  in the patch  $u \neq 0$ , with the identifications

$$\begin{aligned}
a_{1,0} &= 2\sqrt{3}ih_{n+2} \\
a_{2,1} &= -6H_{n+4} \\
a_{4,2} &= f_{8+2n} + 12H_{n+4}^2 \\
a_{4,3} &= f_{8+n} \\
a_{6,4} &= h_{n+2}^2 f_8 - 2H_{n+4} f_{8+n} + g_{12+2n} \\
a_{6,5} &= -2H_{n+4} f_8 + g_{12+n}
\end{aligned} \tag{50}$$

$(a_{4,4} = f_8, a_{6,6} = g_{12})$ .

The relevant sections are

$$\begin{aligned}
a_{1,0} &\in \Gamma(\mathcal{L}^{-5} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{2,1} &\in \Gamma(\mathcal{L}^{-4} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{4,2} &\in \Gamma(\mathcal{L}^{-8} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{6,4} &\in \Gamma(\mathcal{L}^{-6} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^2) \\
a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1),
\end{aligned} \tag{51}$$

which agree with the degrees of Casimirs of  $D_5$  with 0:

$$0, 2, 4, 5, 6, 8 \tag{52}$$

and the coroot expansion:

$$-\theta = 1\alpha_1 + 2\alpha_2 + 2\alpha_3 + 1\alpha_4 + 1\alpha_5. \tag{53}$$

Thus we have derived  $\text{WP}_{(1,1,1,1,2,2)}^5$  from a third-order equation in  $\text{WP}_{(1,1,1,1)}^3$ . This construction of  $D_5$  bundles was not explicitly mentioned in FMW.

### E. $A_4$ bundles ( $A_4$ singularity)

If we further assume

$$\begin{aligned}
f_{8+2n} &= -12H_{n+4}^2 + 12h_{n+2}p_{n+6}, \\
g_{12+2n} &= 12p_{n+6}^2 + 2f_{8+n}H_{n+4} - f_8h_{n+2}^2
\end{aligned} \tag{54}$$



for some  $p_{n+6}$  in (45), we have  $\text{ord}(\Delta) \geq 5$  and the exceptional curve again splits into two lines. In this case, unlike the case for  $E_6$  bundles, the singularity of (45) is not resolved by (46) but we also need to scale  $x'_{\text{new}}$ . This can be done, but we can still use (49) to see which sections are independent. Then (50) reads

$$\begin{aligned}
a_{1,0} &= 2\sqrt{3}ih_{n+2} \\
a_{2,1} &= -6H_{n+4} \\
a_{4,2} &= 12h_{n+2}p_{n+6} \\
a_{4,3} &= f_{8+n} \\
a_{6,4} &= 12p_{n+6}^2 \\
a_{6,5} &= -2H_{n+4}f_8 + g_{12+n},
\end{aligned} \tag{55}$$

where we see that  $a_{4,2}$  and  $a_{6,4}$  are simplified. They are the coefficients of the  $u^2$  term in (49) so using

$$a_{3,2} = 4\sqrt{3}ip_{n+6}, \quad \tilde{Y}'_{\text{new}} = \tilde{Y}' - 2\sqrt{3}iup_{n+6} \tag{56}$$

we have

$$\begin{aligned}
0 &= \tilde{Y}'^2_{\text{new}}v + a_{3,2}\tilde{Y}'_{\text{new}}uv + X'^3 + a_{4,4}X'v^2 + a_{6,6}v^3 \\
&\quad + u \left( a_{1,0}X'\tilde{Y}' + a_{2,1}X'^2 + a_{4,3}X'v + a_{6,5}v^2 \right).
\end{aligned} \tag{57}$$

$a_{4,2}$  and  $a_{6,4}$  in (49) are thus eliminated. In this way, for  $A_4$  bundles, we have obtained a third-order equation in  $\text{WP}^3_{(1,1,1,1)}$  (which is singular but can be smooth by a blow up) with

$$\begin{aligned}
a_{1,0} &\in \Gamma(\mathcal{L}^{-5} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{2,1} &\in \Gamma(\mathcal{L}^{-4} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{3,2} &\in \Gamma(\mathcal{L}^{-3} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{4,3} &\in \Gamma(\mathcal{L}^{-2} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1) \\
a_{6,5} &\in \Gamma(\mathcal{L}^{-0} \otimes (\mathcal{L}^6 \otimes \mathcal{M})^1).
\end{aligned} \tag{58}$$

Again this agrees with the set of Casimirs of  $A_4$  with degrees (with 0):

$$0, 2, 3, 4, 5 \tag{59}$$

and the expansion

$$-\theta = 1\alpha_1 + 1\alpha_2 + 1\alpha_3 + 1\alpha_4. \tag{60}$$

### F. $A_3, A_2, A_1$ bundles ( $D_5, E_6, E_7$ singularity)

So far we have considered bundles for the  $E$  series up to  $E_4 = A_4$ . Since  $E_3$  or  $E_2$  is not a simple Lie algebra, we need a separate discussion for them. Instead, however,  $A_3$ ,  $A_2$  and  $A_1$  bundles can be similarly constructed by setting  $h_{n+2}$ ,  $H_{n+4}$  and  $p_{n+6}$  to zero in this order. In either case, one can show that there is an agreement between the powers of the line bundles and the degrees of the independent Casimirs and the expansion coefficients of the highest weight. Note that in these cases there is still a singularity at  $z = 0$  to be further blown up.

## III. RELATION TO THE INDEPENDENT POLYNOMIALS CHARACTERIZING THE COMPLEX STRUCTURE

In the preceding sections we have seen that for  $E_7$ ,  $E_6$ ,  $D_5$ ,  $A_4$ ,  $A_3$ ,  $A_2$  and  $A_1$  bundles (besides  $E_8$  bundles which are exceptional) the necessary sections which constitute the corresponding weighted projective space stated in Looijenga's theorem are naturally obtained by a series of singularity enhancements of the elliptic manifold followed by the blowing-up procedure. We can now notice that they are nothing but the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory [7, 8] parameterizing the complex structure of the elliptic manifold. The type of the singularity is always the one orthogonal to the gauge group of the vector bundle in the whole  $E_8$ . Indeed, as shown in TABLE I, there is a perfect correspondence between the set of independent polynomials describing the complex structure in 6D and the set of numbers  $d$  and  $s$  characterizing the sections required by Looijenga's theorem, for all the cases of the bundle groups discussed in the preceding section, as well as the other cases for simple, simply-laced gauge groups listed in [8]. As we already noted in the previous footnote 4, a degree  $(an + b)$  polynomial in  $z'$  corresponds to a section of  $\mathcal{L}^{-d} \otimes \mathcal{N}^s$  with  $d = 6a - \frac{b}{2}$  and  $s = a$ .

For  $D_4$  bundles, which are not discussed in the previous section, we consider curves with a  $D_4$  singularity. This can be obtained by restricting  $h_{n+2}$  and  $H_{2n+6}$  to be zero in the  $A_2$  curve (used for  $E_6$  bundles) and requiring the sixth-order term of the discriminant to be of the form [8, 22]

$$4f_{2n+8}^3 + 27g_{3n+12}^2 = j_{n+4}^2 k_{n+4}^2 (j_{n+4}^2 + k_{n+4}^2) \quad (61)$$

for some  $j_{n+4}$  and  $k_{n+4}$ , which are precisely the polynomials with correct degrees needed to constitute the weighted projective space.

$D_6$  and  $A_5$  bundles, which also do not appear in the previous section, are interesting because they are the cases where the singularity has two non-abelian factors. For  $D_6$  bundles, one can show that the relevant curve is, again in the 6D notation,

$$\begin{aligned} 0 = & y^2 + x^3 + 3(h_{2n+4} + h_{n+4}z)x^2 \\ & + z(z + h_n)(p_{2n+8} + q_{n+8}z + s_8z^2)x \\ & + z^2(z + h_n)^2(v_{2n+12} + w_{n+12}z + y_{12}z^2). \end{aligned} \quad (62)$$

This curve has an  $A_1 \times A_1$  ( $SU(2) \times SU(2)$ ) singularity. The lines  $z = 0$  and  $z + h_n = 0$  are the loci of the 7-branes responsible for the two unbroken  $SU(2)$  gauge symmetries. Indeed, the discriminant takes the forms

$$\begin{aligned} \Delta = & z^2 h_n^2 h_{2n+4}^2 K_{4n+16} + O(z^3) \\ = & \tilde{z}^2 h_n^2 \tilde{h}_{2n+4}^2 \tilde{K}_{4n+16} + O(\tilde{z}^3), \end{aligned} \quad (63)$$

where  $\tilde{z} = z + h_n$  and  $\tilde{h}_{2n+4} = h_{2n+4} - h_{n+4}h_n$ .  $K_{4n+16}$  and  $\tilde{K}_{4n+16}$  are given by

$$\begin{aligned} K_{4n+16} = & 9(12h_{2n+4}v_{2n+12} - p_{2n+8}^2), \\ \tilde{K}_{4n+16} = & 9(12\tilde{h}_{2n+4}\tilde{v}_{2n+12} - \tilde{p}_{2n+8}^2), \end{aligned} \quad (64)$$

where  $\tilde{h}_{2n+4}$ ,  $\tilde{v}_{2n+12}$  and  $\tilde{p}_{2n+8}$  are the coefficient polynomials appearing when (62) is re-expressed in terms of  $\tilde{z}$  as

$$\begin{aligned} 0 = & y^2 + x^3 + 3(\tilde{h}_{2n+4} + h_{n+4}\tilde{z})x^2 \\ & + \tilde{z}(\tilde{z} - h_n)(\tilde{p}_{2n+8} + \tilde{q}_{n+8}\tilde{z} + s_8\tilde{z}^2)x \\ & + \tilde{z}^2(\tilde{z} - h_n)^2(\tilde{v}_{2n+12} + \tilde{w}_{n+12}\tilde{z} + y_{12}\tilde{z}^2). \end{aligned} \quad (65)$$

(63) is consistent with the fact that the 6D heterotic charged matter consists of  $n$   $(\mathbf{2}, \mathbf{2})$  and  $4n + 16$   $((\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}))$  computed by the index theorem. Note that the loci of  $h_{2n+4}$  and  $\tilde{h}_{2n+4}$  do not contribute to charged matter since the enhanced fiber type there is  $III$  in the Kodaira classification so the singularity type is unchanged. One can also verify that, in the six-dimensional case, the total number of degrees of freedom of these polynomials

$$(n + 13) + (n + 9) + (n + 5) + (n + 1) + (2n + 13) + (2n + 9) + (2n + 5) - 1 \quad (66)$$

is equal to  $10n + 54$  which precisely matches the number of neutral hypermultiplets. We can see that the sections  $w_{n+12}$ ,  $q_{n+8}$ ,  $h_{n+4}$ ,  $h_n$ ,  $v_{2n+12}$ ,  $p_{2n+8}$  and  $h_{2n+4}$  are precisely the polynomials expected to arise by Looijenga's theorem as are shown in TABLE I.

Similarly, the curve for an  $A_5$  bundle is given by

$$\begin{aligned} 0 = & y^2 + x^3 + 3(h_{n+2}^2 + h_{n+4}z)x^2 \\ & + z(z + h_n)(12h_{n+2}v_{n+6} + q_{n+8}z + s_8z^2)x \\ & + z^2(z + h_n)^2(12v_{n+6}^2 + w_{n+12}z + y_{12}z^2), \end{aligned} \quad (67)$$

which has an  $E_3 = A_2 \times A_1$  ( $SU(3) \times SU(2)$ ) singularity. Here the  $O(z^2)$  term in (63) vanishes ( $K_{4n+16} = 0$  in (64)) and the  $A_1$  singularity at  $z = 0$  is enhanced to  $A_2$ . The discriminant in this case is

$$\begin{aligned} \Delta &= z^3 h_n^2 h_{n+2}^3 K_{4n+18} + O(z^4) \\ &= \tilde{z}^2 h_n^3 \tilde{h}_{2n+4}^2 \tilde{K}_{3n+16} + O(\tilde{z}^3), \end{aligned} \quad (68)$$

being in agreement with the fact that the 6D heterotic charged matter hypermultiplets are  $\frac{n}{2} ((\mathbf{3}, \mathbf{2}) \oplus (\bar{\mathbf{3}}, \mathbf{2}))$ ,  $2n + 9 ((\mathbf{3}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}))$  and  $3n + 16 (\mathbf{1}, \mathbf{2})$ . The number of degrees of freedom of the polynomials also agrees with the number of neutral hypermultiplets  $6n + 37$ . Again, the sections  $w_{n+12}$ ,  $q_{n+8}$ ,  $v_{n+6}$ ,  $h_{n+4}$ ,  $h_{n+2}$  and  $h_n$  have the desired set of  $d$  and  $s$  as are shown in TABLE I.

Finally, let us consider  $E_3 = SU(3) \times SU(2)$  bundles and  $SU(2) \times SU(2)$  bundles. These groups are the orthogonal complements of  $A_5 = SU(6)$  and  $D_6 = SO(12)$  in  $E_8$ . Although these are not simple groups (and hence outside the assumption of Looijenga's theorem), it is interesting to examine whether or not a similar characterization of the bundles is possible in these cases.<sup>5</sup>

For  $E_3 = SU(3) \times SU(2)$  bundles, we consider curves with a  $A_5$  singularity. It is realized by further tuning the complex structure of the  $A_4$  singularity ( $A_4$  bundles) parametrized by

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<sup>5</sup>  $E_2$  contains  $U(1)$  and hence is beyond the scope of this paper.

the polynomials (55) to the following special forms:

$$\begin{aligned}
h_{n+2} &= \tilde{h}_{n+2-r} t_r, \\
H_{n+4} &= \tilde{H}_{n+4-r} t_r, \\
p_{n+6} &= \tilde{h}_{n+2-r} u_{r+4}, \\
f_{n+8} &= \tilde{f}_{n+8-r} t_r - 12 \tilde{H}_{n+4-r} u_{r+4}, \\
g_{n+12} &= 2 \tilde{f}_{n+8-r} u_{r+4} + 2 f_8 H_{n+4}
\end{aligned} \tag{69}$$

for some  $h_{n+2-r}$ ,  $t_r$ ,  $\tilde{H}_{n+4-r}$ ,  $u_{r+4}$  and  $\tilde{f}_{n+8-r}$ , which describes the heterotic configuration with  $4+r$  of  $12+n$  instantons are in  $SU(2)$  in  $E_3$  and the remaining  $8+n-r$  are in  $SU(3)$ . Apparently, besides  $f_8$  which describes the complex structure of the heterotic Calabi-Yau manifold, these five sections are needed to parametrize the moduli space of the bundle. However, defining

$$\begin{aligned}
p_{n+6} &\equiv \tilde{h}_{n+2-r} u_{r+4}, \\
f_{n+8}^{(1)} &\equiv \tilde{f}_{n+8-r} t_r, \\
f_{n+8}^{(2)} &\equiv \tilde{H}_{n+4-r} u_{r+4}, \\
g'_{n+12} &\equiv 2 \tilde{f}_{n+8-r} u_{r+4},
\end{aligned} \tag{70}$$

(69) can be formally written as

$$\begin{aligned}
h_{n+2} &= \frac{2 p_{n+6} f_{n+8}^{(1)}}{g'_{n+12}}, \\
H_{n+4} &= \frac{2 f_{n+8}^{(1)} f_{n+8}^{(2)}}{g'_{n+12}}, \\
p_{n+6} &= p_{n+6}, \\
f_{n+8} &= f_{n+8}^{(1)} - 12 f_{n+8}^{(2)}, \\
g_{n+12} &= g'_{n+12} + 2 f_8 H_{n+4}
\end{aligned} \tag{71}$$

( $2 f_8 H_{n+4}$  can be absorbed in  $g_{n+12}$  by redefinition). Therefore, provided that  $2 p_{n+6} f_{n+8}^{(1)}$  and  $2 f_{n+8}^{(1)} f_{n+8}^{(2)}$  are divisible by  $g'_{n+12}$ , they are parametrized by the four independent combinations  $p_{n+6}$ ,  $f_{n+8}^{(1)}$ ,  $f_{n+8}^{(2)}$  and  $g'_{n+12}$ . The corresponding set of  $d$  and  $s$  are then 3, 2, 2, 0 and 1, 1, 1, 1, respectively. Thus we have seen that, though non-simple, the  $E_3$  bundle is also parametrized by the sections specified by the Casimirs of  $A_2 = SU(3)$  and  $A_1 = SU(2)$ , which are  $\{3, 2\}$  and  $\{2\}$ , and the coroot expansion coefficients  $-\theta = \alpha_1 + \alpha_2$  and  $-\theta = \alpha_1$ .

For  $SU(2) \times SU(2)$  bundles, the relevant curve is the one with a  $D_6$  singularity. Such a curve is realized by setting

$$\tilde{h}_{n+2-r} = 0 \tag{72}$$

in the  $A_5$  curve (69). Consequently,  $p_{n+6} = 0$  in (70), so that the moduli space of  $SU(2) \times SU(2)$  bundle is parametrized by  $f_{n+8}^{(1)}$ ,  $f_{n+8}^{(2)}$  and  $g'_{n+12}$ . The corresponding set of  $d$  and  $s$  are 2, 2, 0 and 1, 1, 1, respectively. These agree with the Casimirs and the coroot expansion coefficients of the two  $SU(2)$ 's.

#### IV. WHY SHOULD THIS BE SO? : THE MORDELL-WEIL LATTICE

In the previous sections we have seen that the sections of a particular set of line bundles coordinatizing Looijenga's weighted projective spaces can be automatically obtained as the coefficients of curves arising from a series of blow-ups in  $dP_9$ . They can be thought of as the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory parameterizing the complex structure of the elliptic manifold, in which the gauge group of the bundle and the singularity are orthogonal to each other in  $E_8$ . In this section we explain why this is so.

As we stated in the previous section, the  $dP_9$  we have blown up is supposed to be a half of a  $K3$  in the stable degeneration limit, and the values of sections at infinity determine the spectral cover of the dual heterotic string theory.

Physically, a spectral cover describes the Wilson lines in the elliptic fibers of the heterotic Calabi-Yau over which the vector bundle is defined. Therefore, if the algebra of the Wilson lines is  $H$ , the Lie algebra of the unbroken gauge subgroup  $G$  is the commutant of  $H$  in  $E_8$ . Thus it is natural to derive  $H$  bundles when the singularity of  $dP_9$  is  $G$ . This is a “physical” explanation, but there must also be a pure “mathematical” explanation which accounts for why the series of vector bundles are derived by the series of blow-ups, without referring to the heterotic/F-theory duality. What makes it possible is the structure theorem of the Mordell-Weil lattice.

The Mordell-Weil lattice [5] is the Mordell-Weil group [11] equipped with a certain bilinear form. The Mordell-Weil group  $E(K)$  of a rational elliptic surface ( $=dP_9$ ) is defined as an Abelian group of rational sections of  $dP_9$ , where  $K$  is the field of rational functions of the

coordinate  $z$  of the base  $\mathbb{P}^1$  of  $dP_9$ . The addition of two sections is defined by the addition rule on an elliptic curve applied fiberwise, that is, as the addition of the two arguments of the  $\wp$  (and also  $\wp'$ ) function parameterizing the two sections. As is well known, the argument variable inside  $\wp$  (and  $\wp'$ ) is nothing but the complex coordinate itself if the fiber torus is expressed as a parallelogram with the two sets of sides identified.

$E(K)$  is called the Mordell-Weil lattice [12] if it is endowed with a bilinear form, or a height pairing,  $(P, Q)$  for sections  $P, Q \in E(K)$  such that<sup>6</sup>

$$(P, Q) = P \cdot O + Q \cdot O - P \cdot Q + 1 - \sum_{v \in \text{singularities}} \text{contr}_v(P, Q), \quad (73)$$

$$(P, P) = 2 + 2P \cdot O - \sum_{v \in \text{singularities}} \text{contr}_v(P, P), \quad (74)$$

where  $\cdot$  denotes the intersection pairing. For each singularity  $v$ , the function  $\text{contr}_v$  of a pair of sections  $P, Q$  is defined as

$$\text{contr}_v(P, Q) = \begin{cases} 0 & \text{if } i(P) = 0 \text{ or } i(Q) = 0, \\ (C_v^{-1})_{i(P), i(Q)} & \text{otherwise,} \end{cases} \quad (75)$$

where  $C_v$  is the Cartan matrix corresponding to the singularity  $v$ , and  $i(P)$  ( $i(Q)$ ) is either of  $0, 1, \dots, \text{rank} C_v$  labeling the fiber component of  $v$  which (uniquely) intersects with the section  $P$  ( $Q$ ). The fiber labeled as “the zeroth” ( $i = 0$ ) is the one that intersects with the zero section.

One of the remarkable results of [5] is that then  $E(K)$  is roughly the orthogonal complement of the singularity in the  $E_8$  root lattice. More precisely [5],

$$E(K) \simeq L^* \otimes (T'/T), \quad (76)$$

where  $T$  is the singularity lattice embedded into the  $E_8$  root lattice  $\Lambda_{E_8}$ ,  $L$  is its orthogonal lattice with respect to the specified embedding into  $\Lambda_{E_8}$ ,  $L^*$  is the dual of  $L$ , and

$$T' = T \otimes \mathbb{Q} \cap \Lambda_{E_8}. \quad (77)$$

This is a geometrical manifestation of the fact that if the instanton is in the group  $H$ , the unbroken gauge group is the commutant of  $H$  in  $E_8$ . By this theorem we can now explain

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<sup>6</sup> The fact that the arithmetic genus of  $dP_9$  is one is taken into account here.

why we could derive  $E_N$  bundles by blowing up the  $A_{9-N}$  singularities: As we mentioned earlier, an  $E_N$  bundle is constructed from the spectral cover, whose equation determines as the intersections with the elliptic fiber at infinity the Wilson lines of the vector bundle. As one can check explicitly, these intersection points are extended into sections in the  $dP_9$  [4, 13], obtaining the  $E_N$  weight lattice generated by the sections. The structure theorem of the Mordell-Weil lattice then tells us that this occurs precisely when the singularity lattice is the orthogonal complement of the  $E_N$  weight lattice, which is  $A_{9-N}$ .

We should mention that the rational elliptic surfaces with various sections and singularities are known to be identified [14–20] as the total spaces of Seiberg-Witten curves for the four-, five- and six-dimensional so-called  $E_N$  theories [21], where the  $u$  parameter becomes the coordinate of the  $\mathbb{P}^1$  base. Indeed, the curves we considered in section 2 are exactly the same as the ones found in [15, 17], although the line bundles of the sections and their relation to Looijenga’s weighted projective spaces were not investigated there. We also note that the values of sections at infinity are known to determine the mass parameters of the gauge theory whose Seiberg-Witten curve (together with the  $u$ -plane ( $\mathbb{P}^1$ )) is a rational elliptic surface allowing those sections.

The Mordell-Weil lattice also provides us with an understanding of the relation between the singularity and the occurrence of chiral matter in F-theory. (This fact was already observed and briefly mentioned in [6].) In the standard explanation for the chiral matter generation [9], one considers an enhanced singularity [7, 8], at which the light membrane (in the M-theory dual picture) wrapping the extra shrinking two-cycle is identified as the origin of the chiral matter. On the other hand, it was shown by using the Leray spectral sequence [2, 3] that chiral matter is localized where one or some of the sections of  $dP_9$  goes to the zero section. Again, the relation between the two pictures of matter generation may also be understood as a consequence of the structure theorem of the Mordell-Weil lattice. Indeed, the theorem says if some of the sections disappear in  $dP_9$ , then the singularity lattice, which is the orthogonal complement in  $E_8$ , becomes larger, leading to a singularity enhancement. Also, in view of the isomorphism between the string junction algebra and the Picard lattice of a rational elliptic surface [18], it gives support to the understanding of matter generation in F-theory in terms of string junctions [6, 22, 23].



## V. CONCLUSIONS

We have shown that the holomorphic vector bundles for gauge groups  $E_N$  ( $N = 4, \dots, 8$ ) and  $A_n$  ( $n = 1, 2, 3$ ) can be obtained systematically by a series of blowing-ups in the rational elliptic surface according to Tate's algorithm. The sections of correct line bundles claimed to arise by Looijenga's theorem have been found automatically by this procedure. We have also pointed out that the sections parameterizing a Looijenga's weighted projective space are nothing but the four-dimensional analogue of the set of independent polynomials in the six-dimensional F-theory parameterizing the complex structure of the elliptic manifold with a singularity orthogonal to the gauge group of the vector bundle in the whole  $E_8$ . We have explained the reason for this by using the structure theorem of the Mordell-Weil lattice. We have also used it to elucidate the relation between the singularity and the occurrence of chiral matter in F-theory.

The Mordell-Weil lattice is classified into 74 different patterns of decompositions of the  $E_8$  root lattice, of which we have used only a few in this paper. It would be interesting to extend the analysis to other cases in which the Mordell-Weil group has a torsion [11]. Also, in confirming the relation between the sections and the independent polynomials in section 3, we have derived some examples of curves which possess two factorized singularities leading to an unbroken gauge symmetry of a direct product of two simple groups. A thorough investigation of this type of curves is in progress and will be reported elsewhere.

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TABLE I:

Bundle gauge group ( $= H$ )	Singularity ( $= G$ )	6D neutral matter	Independent polynomial	$d$	$s$
$E_7$	$A_1$	$(18n + 83)\mathbf{1}$	$g_{12+n}$	0	1
			$f_{8+n}$	2	1
			$g_{12+2n}$	6	2
			$f_{8+2n}$	8	2
			$h_{2n+4}$	10	2
			$g_{12+3n}$	12	3
			$f_{8+3n}$	14	3
			$g_{12+4n}$	18	4
$E_6$	$A_2$	$(12n + 66)\mathbf{1}$	$g_{12+n}$	0	1
			$f_{8+n}$	2	1
			$h_{n+2}$	5	1
			$g_{12+2n}$	6	2
			$f_{8+2n}$	8	2
			$H_{2n+6}$	9	2
			$g_{12+3n}$	12	3
$D_5$	$A_3$	$(8n + 51)\mathbf{1}$	$g_{12+n}$	0	1
			$f_{8+n}$	2	1
			$H_{n+4}$	4	1
			$h_{n+2}$	5	1
			$g_{12+2n}$	6	2
			$f_{8+2n}$	8	2
$A_4$	$A_4$	$(5n + 36)\mathbf{1}$	$g_{12+n}$	0	1
			$f_{8+n}$	2	1
			$p_{n+6}$	3	1
			$H_{n+4}$	4	1
			$h_{n+2}$	5	1

(Cont'd)

$A_3$	$D_5$	$(4n + 33)\mathbf{1}$	$g_{n+12}$	0	1
			$f_{n+8}$	2	1
			$p_{n+6}$	3	1
			$H_{n+4}$	4	1
$A_2$	$E_6$	$(3n + 28)\mathbf{1}$	$g_{n+12}$	0	1
			$f_{n+8}$	2	1
			$p_{n+6}$	3	1
$A_1$	$E_7$	$(2n + 21)\mathbf{1}$	$g_{n+12}$	0	1
			$f_{n+8}$	2	1
$D_6$	$A_1 \oplus A_1$	$(10n + 54)\mathbf{1}$	$w_{n+12}$	0	1
			$q_{n+8}$	2	1
			$h_{n+4}$	4	1
			$h_n$	6	1
			$v_{2n+12}$	6	2
			$p_{2n+8}$	8	2
			$h_{2n+4}$	10	2
$A_5$	$A_2 \oplus A_1$	$(6n + 37)\mathbf{1}$	$w_{n+12}$	0	1
			$q_{n+8}$	2	1
			$v_{n+6}$	3	1
			$h_{n+4}$	4	1
			$h_{n+2}$	5	1
			$h_n$	6	1
$D_4$	$D_4$	$(6n + 44)\mathbf{1}$	$g_{n+12}$	0	1
			$f_{n+8}$	2	1
			$j_{n+4}$	4	1
			$k_{n+4}$	4	1
			$g_{2n+12}$	6	2